A remark on Schwarz's topological field theory

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Abstract

The standard evaluation of the partition function Z of Schwarz's topological field theory results in the Ray–Singer analytic torsion. Here we present an alternative evaluation which results in $Z = 1$. Mathematically, this amounts to a novel perspective on analytic torsion: it can be formally written as a ratio of volumes of spaces of differential forms which is formally equal to 1 by Hodge duality. An analogous result for Reidemeister combinatorial torsion is also obtained.

1 Introduction

Analytic torsion [1] arises in a quantum field theoretic context as (the square of) the partition function of Schwarz's topological field theory [2, 3] (see [4] for a detailed review). This has turned out to be an important result in topological quantum field theory; for example it is used to evaluate the semiclassical approximation for the Chern–Simons partition function [5, 6], which gives a QFT-predicted formula for an asymptotic limit of the Witten–Reshetikhin–Turaev 3-manifold invariant [7] since this invariant arises as the partition function of the Chern–Simons gauge theory on the 3-manifold [5]. See also [8] for a review of Schwarz's topological field theory in a general context, and [9] for some explicit results in the case of hyperbolic 3-manifolds.

The partition function, Z , of Schwarz's topological field theory is a priori a formal, mathematically ill-defined quantity and its evaluation [2, 3, 4] is by formal manipulations which in the end lead to a mathematically meaningful result: $Z=\tau^{1/2}$ where τ is the analytic torsion of the background manifold. In this paper we show (§2) that there is an alternative formal evaluation of the partition function which results in the trivial answer $Z=1$. This result amounts to a novel perspective on analytic torsion: we find that it can be formally written as a certain ratio of volumes of spaces of differential forms which is formally equal to 1 by Hodge duality.

Reidemeister combinatorial torsion (R-torsion) [10, 1] arises as the partition function of a discrete version of Schwarz's topological field theory [11, 12]. This is of potential interest if one is to attempt to capture the invariants of topological QFT in a discrete, i.e. combinatorial, setting. In §3 an analogue of the above-mentioned result is derived for combinatorial torsion.

2 Schwarz's topological field theory and analytic torsion

We begin by recalling the evaluation of the partition function

$$Z=\frac{1}{V}\int\mathcal{D}\omega\,e^{-S(\omega)} \tag{2.1}$$

of Schwarz's topological field theory [2, 3, 4]. Here V is a normalisation factor to be specified below. The background manifold ("spacetime") M is closed, oriented, riemannian, and has odd dimension $n=2m+1$. For simplicity we assume m is odd; then the following variant of Schwarz's topological field theory can be considered [4]: The field $\omega\in\Omega^m(M,E)$ is an m-form on M with values in some flat $O(N)$ vectorbundle E over M. The action functional is

$$S(\omega)=\int_M\omega\wedge d_m\omega\,. \tag{2.2}$$

Here $d_p : \Omega^p \to \Omega^{p+1}$ ($\Omega^p \equiv \Omega^p(M,E)$) is the exterior derivative twisted by a flat connection on E (which we suppress in the notation), and a sum over vector indices is implied in (2.2) [1]. A choice of metric on M determines an inner product in each Ω^p , given in terms of the Hodge operator $*$ by

$$\langle \omega \, , \omega' \rangle = \int_M \omega \wedge *\omega' \tag{2.3}$$

Using this, the action (2.2) can be written as $S(\omega) = \langle \omega \, , *d_m \omega \rangle$. Let $\ker(S)$ denote the radical of the quadratic functional S and $\ker(d_p)$ the nullspace of d_p. Then $\ker(S) = \ker(d_m)$, and after decomposing the integration space in (2.1) as $\Omega^m = \ker(S) \oplus \ker(S)^\perp$ the partition function can be formally evaluated to get

$$Z = \frac{V(\ker(S))}{V} \det{}'((*d_m)^2)^{-1/4} = \frac{V(\ker(S))}{V} \det{}'(d_m^* d_m)^{-1/4} \tag{2.4}$$

(we are ignoring certain phase and scaling factors; see [13] for these). Here $V(\ker(S))$ denotes the formal volume of $\ker(S)$. The obvious normalisation choice $V = V(\ker(S))$ does not preserve a certain symmetry property which the partition function has when S is non-degenerate [4]; therefore we do not use this but instead proceed, following Schwarz, by introducing a resolvent for S. For simplicity we assume that the cohomology of d vanishes, i.e. $\mathrm{Im}(d_p) = \ker(d_{p+1})$ for all p ($\mathrm{Im}(d_p)$ is the image of d_p). Then S has the resolvent

$$0 \longrightarrow \Omega^0 \xrightarrow{d_0} \Omega^1 \xrightarrow{d_1} \ldots \longrightarrow \Omega^{m-1} \xrightarrow{d_{m-1}} \ker(S) \longrightarrow 0 \tag{2.5}$$

which we use in the following to formally rewrite $V(\ker(S))$. The orthogonal decompositions

$$\Omega^p = \ker(d_p) \oplus \ker(d_p)^\perp \tag{2.6}$$

give the formal relations

$$V(\Omega^p) = V(\ker(d_p))\, V(\ker(d_p)^\perp) \, . \tag{2.7}$$

[1] Note that (2.2) vanishes if m is even.

The maps d_p restrict to isomorphisms $d_p : \ker(d_p)^{\perp} \xrightarrow{\cong} \ker(d_{p+1})$, giving the formal relations

$$V(\ker(d_{p+1})) = |\det{}'(d_p)|\, V(\ker(d_p)^{\perp})\,. \tag{2.8}$$

Combining (2.7)–(2.8) we get

$$V(\ker(d_{p+1})) = \det{}'(d_p^* d_p)^{1/2}\, V(\Omega^p)\, V(\ker(d_p))^{-1}\,. \tag{2.9}$$

Now a simple induction argument based on (2.9) and starting with $V(\ker(S)) = V(\ker(d_m))$ gives the formal relation

$$V(\ker(S)) = \prod_{p=0}^{m-1} \Big(\det{}'(d_p^* d_p)^{1/2}\, V(\Omega^p) \Big)^{(-1)^p}\,. \tag{2.10}$$

A natural choice of normalisation is now [2]

$$V = \prod_{p=0}^{m-1} V(\Omega^p)^{(-1)^p}\,. \tag{2.11}$$

Substituting (2.10)–(2.11) in (2.4) gives

$$Z = \left[\prod_{p=0}^{m-1} \det{}'(d_p^* d_p)^{\frac{1}{2}(-1)^p} \right] \det{}'(d_m^* d_m)^{-1/4}\,. \tag{2.12}$$

These determinants can be given well-defined meaning via zeta-regularisation [1], resulting in a mathematically meaningful expression for the partition function. As a simple consequence of Hodge duality we have $\det'(d_p^* d_p) = \det{}'(d_{n-p-1}^* d_{n-p-1})$, which allows to rewrite (2.12) as

$$Z = \tau(M\,, d)^{1/2} \tag{2.13}$$

where

$$\tau(M\,, d) = \prod_{p=0}^{n-1} \det{}'(d_p^* d_p)^{\frac{1}{2}(-1)^p}\,. \tag{2.14}$$

[2] This choice can be motivated by the fact that, in an analogous finite-dimensional setting, the partition function then continues to exhibit a certain symmetry property which it has when S is non-degenerate [4].

This is the Ray–Singer analytic torsion [1]; it is independent of the metric, depending only on M and d. This variant of Schwarz's result is taken from [4]; it has the advantage that the resolvent (2.5) is relatively simple. The cases where m need not be odd, and the cohomology of d need not vanish, are covered in [2, 3] (see also [4] for the latter case). Everything we do in the following has a straightforward extension to these more general settings, but for the sake of simplicity and brevity we have omitted this.

We now proceed to derive a different answer for Z to the one above. Our starting point is (2.13)–(2.14) which we consider as a formal expression for Z, i.e. we do not carry out the zeta regularisation of the determinants. Instead, we use (2.8) to formally write

$$\det{}'(d_p^* d_p)^{1/2} = \frac{V(\ker(d_{p+1}))}{V(\ker(d_p)^{\perp})} \tag{2.15}$$

Substituting this in (2.14) and using (2.7) we find [3]

$$\tau(M\,,d) = \frac{V(\Omega^1)\,V(\Omega^3)\ldots V(\Omega^n)}{V(\Omega^0)\,V(\Omega^2)\ldots V(\Omega^{n-1})} \tag{2.16}$$

Formally, the ratio of volumes on the r.h.s. equals 1 due to

$$V(\Omega^p) = V(\Omega^{n-p})\,, \tag{2.17}$$

which is a formal consequence of the Hodge star operator being an orthogonal isomorphism from Ω^p to Ω^{n-p}. (Recall $\langle *\omega\,,*\omega'\rangle = \langle\omega\,,\omega'\rangle$ for all $\omega\,,\omega' \in \Omega^p$.) This implies $Z = 1$ due to (2.13).

The formal relation (2.16) shows that analytic torsion can be considered as a "volume ratio anomaly": The ratio of the volumes on the r.h.s. of (2.16) is formally equal to 1, but when $\tau(M\,,d)$ is given well-defined meaning via zeta regularisation of (2.14) a non-trivial value results in general.

It is also interesting to consider the case where n is even: In this case, using (2.7)–(2.8) we get in place of (2.16) the formal relation

$$\frac{V(\Omega^0)\,V(\Omega^2)\ldots V(\Omega^n)}{V(\Omega^1)\,V(\Omega^3)\ldots V(\Omega^{n-1})} = \prod_{p=0}^{n-1} \det{}'(d_p^* d_p)^{\frac{1}{2}(-1)^p} = 1 \tag{2.18}$$

[3] This relation is obtained without any restriction on m, i.e. for arbitrary odd n.

The last equality is an easy consequence of Hodge duality and continues to hold after the determinants are given well-defined meaning via zeta regularisation [1]. On the other hand, the ratio of volumes on the l.h.s. is no longer formally equal to 1 by Hodge duality.

3 The discrete analogue

Given a simplicial complex K triangulating M a discrete version of Schwarz's topological field theory can be constructed which captures the topological quantities of the continuum theory [11, 12]. The discrete theory uses $\widehat{K}$, the cell decomposition dual to K, as well as K itself. This necessitates a field doubling in the continuum theory prior to discretisation: An additional field ω' is introduced and the original action $S(\omega) = \langle \omega\,, *d_m\omega \rangle$ is replaced by the doubled action,

$$\widetilde{S}(\omega,\omega') = \Big\langle \begin{pmatrix} \omega \\ \omega' \end{pmatrix}, \begin{pmatrix} 0 & *d_m \\ *d_m & 0 \end{pmatrix} \begin{pmatrix} \omega \\ \omega' \end{pmatrix} \Big\rangle = 2\int_M \omega' \wedge d_m\omega\,. \tag{3.1}$$

This theory (known as the abelian BF theory [8]) has the same topological content as the original one; in particular its partition function, $\widetilde{Z}$, can be evaluated in an analogous way to get $\widetilde{Z} = Z^2 = \tau(M,d)$. The discretisation prescription is now [11, 12]:

$$(\omega,\omega') \;\to\; (\alpha,\alpha') \in C^m(K) \times C^m(\widehat{K}) \tag{3.2}$$

$$\widetilde{S}(\omega,\omega') \;\to\; \widetilde{S}(\alpha,\alpha') = \Big\langle \begin{pmatrix} \alpha \\ \alpha' \end{pmatrix}, \begin{pmatrix} 0 & *^{\widehat{K}} d_m^{\widehat{K}} \\ *^K d_m^K & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \alpha' \end{pmatrix} \Big\rangle \tag{3.3}$$

Here $C^p(K) = C^p(K,E)$ is the space of p-cochains on K with values in the flat $O(N)$ vectorbundle E and $d_p^K : C^p(K) \to C^{p+1}(K)$ is the coboundary operator twisted by a flat connection on E , with $C^q(\widehat{K})$ and $d_q^{\widehat{K}}$ being the corresponding $\widehat{K}$ objects; $*^K : C^p(K) \to C^{n-p}(\widehat{K})$ and $*^{\widehat{K}} : C^q(\widehat{K}) \to C^{n-q}(K)$ are the duality operators induced by the duality between p-cells of K and $(n-p)$-cells of $\widehat{K}$. The p-cells of K and $\widehat{K}$ determine canonical inner products in $C^p(K)$ and $C^p(\widehat{K})$ for each p, and with respect to these $*^K$ and $*^{\widehat{K}}$ are orthogonal maps. (The definitions and background can be found in [14]; see also [1] and [11].) As in §2 we are assuming that m is odd and that the cohomology of the flat connection on E vanishes: $H^*(M,E)=0$. Then the partition function of the discrete theory, denoted $\widetilde{Z}_K$, can be evaluated by formal manipulations analogous to those in §2 (see [11, 12]) and the resulting expression can be written as either

$$\widetilde{Z}_K = \tau(K, d^K) \qquad \text{or} \qquad \widetilde{Z}_K = \tau(\widehat{K}, d^{\widehat{K}}) \tag{3.4}$$

where

$$\tau(K, d^K) = \prod_{p=0}^{n-1} \det{}'(\partial_{p+1}^K d_p^K)^{\frac{1}{2}(-1)^p} \tag{3.5}$$

and $\tau(\widehat{K}, d^{\widehat{K}})$ is defined analogously. Here ∂_{p+1}^K denotes the adjoint of d_p^K (it can be identified with the boundary operator on the $(p+1)$-chains of K). The quantities $\tau(K, d^K)$ and $\tau(\widehat{K}, d^{\widehat{K}})$ coincide; in fact (3.5) is the Reidemeister combinatorial torsion of M determined by the given flat connection on E, and is the same for all cell decompositions K of M [10, 1]. (This is analogous to the metric-independence of analytic torsion.) Moreover, the analytic and combinatorial torsions coincide [15], so the discrete partition function in fact reproduces the continuum one:

$$\widetilde{Z}_K = \widetilde{Z}\,. \tag{3.6}$$

We now present an analogue of the formal argument which led to $Z=1$ in §2. Consider

$$\tau(K, d^K)\, \tau(\widehat{K}, d^{\widehat{K}}) = \prod_{p=0}^{n-1} \det{}'(\partial_{p+1}^K d_p^K)^{\frac{1}{2}(-1)^p} \det{}'(\partial_{p+1}^{\widehat{K}} d_p^{\widehat{K}})^{\frac{1}{2}(-1)^p}\,. \tag{3.7}$$

Using the analogues of (2.15) and (2.7) in the present setting,

$$\det{}'(\partial_{p+1}^K d_p^K)^{1/2} = \frac{V(\ker(d_{p+1}^K))}{V(\ker(d_p^K)^{\perp})} \tag{3.8}$$

and

$$V(C^p(K)) = V(\ker(d_p^K))\, V(\ker(d_p^K)^{\perp})\,, \tag{3.9}$$

and the corresponding $\widehat{K}$ relations, we find an analogue of the formal relation (2.16):

$$\begin{aligned} &\tau(K, d^K)\, \tau(\widehat{K}, d^{\widehat{K}}) \\ &= \frac{V(C^1(K))\, V(C^3(K)) \ldots V(C^n(K))}{V(C^0(K))\, V(C^2(K)) \ldots V(C^{n-1}(K))} \, \frac{V(C^1(\widehat{K}))\, V(C^3(\widehat{K})) \ldots V(C^n(\widehat{K}))}{V(C^0(\widehat{K}))\, V(C^2(\widehat{K})) \ldots V(C^{n-1}(\widehat{K}))} \end{aligned} \tag{3.10}$$

Formally, the r.h.s. equals 1 due to

$$V(C^p(K)) = V(C^{n-p}(\widehat{K}))\,, \tag{3.11}$$

which is a formal consequence of the duality operator being an orthogonal isomorphism from $C^p(K)$ to $C^{n-p}(\widehat{K})$ (i.e. $\langle *^K\alpha\,, *^K\alpha'\rangle = \langle\alpha\,,\alpha'\rangle$ for all $\alpha, \alpha' \in C^p(K)$). This implies that, formally,

$$\widetilde{Z}_K = \Big(\tau(K, d^K)\,\tau(\widehat{K}, d^{\widehat{K}})\Big)^{1/2} = 1\,. \tag{3.12}$$

Thus we see that combinatorial torsion can also be considered as a "volume ratio anomaly" in an analogous way to analytic torsion.

Finally, in the n even case it is straightforward to find a combinatorial analogue of the formal relation (2.18) –we leave this to the reader.

Acknowledgements. D.A. is supported by an ARC fellowship. E.P. is supported by funding from FORBAIRT and Trinity College Dublin.

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